

Equivalence groupoids of classes of linear ordinary differential equations and their group classification

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Admissible point transformations of classes of r th order linear ordinary differential equations (in particular, the whole class of such equations and its subclasses of equations in the rational and Laguerre–Forsyth canonical forms) are exhaustively described. Using these results the group classification of such equations is carried out within the algebraic approach in three different ways.

Key words: equivalence group; equivalence groupoid; Lie symmetry; group classification of differential equations; linear ordinary differential equation; normalized class of differential equations

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1 Introduction

The study of Lie symmetries of ordinary differential equations (ODEs) has a long history and “Lie theory” was just started as a systematical and elegant approach to integration of various classes of ODEs. The first results on possible dimensions of the maximal Lie invariance algebras of ODEs of any fixed order were obtained by S. Lie, see, e.g., [17, pp. 294–301]. Namely, S. Lie proved that the dimension of the maximal Lie invariance algebra of an r th order ODE is infinite for $r = 1$, not greater than 8 for $r = 2$ and not greater than $r + 4$ for $r \geq 3$. He also showed that, for equations of the order $r \geq 2$, the maximal dimension of the invariance algebra is reached for equations that are reduced by point transformations to the elementary equation $x^{(r)} = 0$.

In spite of the fact that group properties of r th order linear ODEs are well known (see, e.g., detailed reviews [11, 18, 29] and textbooks [12, 23, 30]), in the present paper we study them from another side, involving the complete description of the whole set of admissible transformations between such equations. This approach is quite effective for solving group classification problems in classes of partial differential equations: see, e.g., [3, 25, 27, 28, 33] and references therein. Previously, in [13, 19], the group classification of linear ODEs was carried out within the framework of the classical infinitesimal approach, which led to cumbersome calculations. Although this approach is the most commonly used in group analysis of differential equations, it is efficient only for classes of simple structure. In [23, pp. 217–218] the solution of the group classification problem of linear ODEs is related to Wilczynski’s result [37] on relative invariants of the Laguerre–Forsyth canonical form of these equations.

Thus, the purpose of this paper is to carry out the complete group classification of the class \mathcal{L} of r th order ($r \geq 2$) linear ODEs using subalgebra analysis of the equivalence algebra associated with \mathcal{L} . This properly works since the class \mathcal{L} is (pointwise) normalized, i.e., transformations from its point equivalence group¹ G^\sim generate all admissible point transformations between equations from \mathcal{L} . Note that the equivalence group G^\sim of the class \mathcal{L} was first found

¹There exist other names for this notions, e.g. “structure invariance group” [30].

by Stäckel [31]². The set of admissible transformations of any class of differential equations possesses the groupoid structure and is called the *equivalence groupoid* of this class [3, 25]. See, e.g., [3, 25, 27, 34] for the definition of normalized classes and other related notions. So, we can say that the equivalence groupoid of \mathcal{L} is generated by the equivalence group G^\sim .

In Section 2 we begin the study of the class \mathcal{L} of r th order ($r \geq 2$) linear ODEs with the description of its equivalence groupoid in terms its usual equivalence group and normalization. As a rule, we present only the components of equivalence transformations that correspond to the dependent and independent variables. We can gauge arbitrary elements of the class \mathcal{L} by parameterized families of transformations from G^\sim , which induce mappings of the class \mathcal{L} onto its subclasses. Two such gauges for arbitrary elements related to the subleading-order derivatives are well known. They result in the rational canonical form (the subclass \mathcal{L}_1) and Laguerre–Forsyth canonical form (the subclass \mathcal{L}_2). It appears that both the subclasses \mathcal{L}_1 and \mathcal{L}_2 are also normalized with respect to their usual equivalence groups. Then we study two gauges for arbitrary elements related to the lowest-order derivatives, which give the first and second Arnold canonical forms. The corresponding subclasses are not normalized in the usual sense and hence these gauges are not convenient for symmetry analysis. Having the chain of nested normalized classes $\mathcal{L} \supset \mathcal{L}_1 \supset \mathcal{L}_2$, we can classify Lie symmetries of r th order linear ODEs within the algebraic approach in three different ways, which is done in Section 3. In the final section we summarize results of the paper and discuss problems and directions for further investigation.

2 Equivalence groupoids of classes of linear ODEs

Consider the class \mathcal{L} of r th order linear ODEs, which have the form

$$x^{(r)} + a_{r-1}(t)x^{(r-1)} + \cdots + a_1(t)x^{(1)} + a_0(t)x = b(t), \quad (1)$$

where a_{r-1}, \dots, a_1, a_0 and b are arbitrary smooth functions of t , $x = x(t)$ is the unknown function, $x^{(k)} = d^k x / dt^k$, $k = 1, \dots, r$, $r \geq 2$. Below we also use the notation $x' = dx/dt$ and $x'' = d^2 x / dt^2$ for the first and second derivatives, respectively. We assume that all variables, functions and other values are either real or complex, i.e., the underlying field \mathbb{F} is either $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, respectively. We work within the local approach.

There are two different cases for the structure of the equivalence groupoid of the entire class (1) depending on the value r , namely $r = 2$ and $r \geq 3$. We begin with the case $r = 2$.

Proposition 1. *The equivalence group G^\sim of the class (1) with $r = 2$ consists of the transformations whose projections to the variable space have the form*

$$\tilde{t} = T(t), \quad \tilde{x} = X^1(t)x + X^0(t), \quad (2)$$

where T , X^1 and X^0 are arbitrary smooth functions of t with $T_t X^1 \neq 0$.

Proof. Suppose that a point transformation \mathcal{T} of the general form $\tilde{t} = T(t, x)$, $\tilde{x} = X(t, x)$, where $J = |\partial(T, X)/\partial(t, x)| \neq 0$, connects two fixed second-order linear ODEs \mathcal{E} and $\tilde{\mathcal{E}}$. We substitute the expressions for the new variables (which are with tildes) and the corresponding derivatives in terms of the old variables (which are without tildes) into $\tilde{\mathcal{E}}$. The equality obtained should be identically satisfied on solutions of the equation \mathcal{E} . Therefore, additionally substituting the expression for x'' implied by the equation \mathcal{E} , we can split the equality with respect to the derivative x' . Collecting the coefficients of $(x')^3$, we obtain the equation

$$X_{xx}T_x - X_xT_{xx} + \tilde{a}_1X_xT_x^2 + (\tilde{a}_0X - \tilde{b})T_x^3 = 0.$$

²Recall also the contribution by Halphen [10], Laguerre [14], Forsyth [7] and Wilczynski [37] in this subject.

As \tilde{a}_1 , \tilde{a}_0 and \tilde{b} are the only arbitrary elements involved in this equation and we study the equivalence group, we can vary the arbitrary elements and can hence split with respect to them. Therefore, $T_x = 0$. Then the terms with $(x')^2$ give the equation $T_t X_{xx} = 0$. As $J \neq 0$, we have that $T = T(t)$ and $X = X^1(t)x + X^0(t)$. After the additional splitting with respect to x , the other determining equations define the transformation components for the arbitrary elements. \square

Proposition 2. *The equivalence groupoid of the class \mathcal{L} of second-order linear ODEs is generated by transformations from the equivalence group G^\sim of this class and transformations from the point symmetry group of the equation $x'' = 0$. Therefore, the class \mathcal{L} is semi-normalized in the usual sense.*

Proof. It is commonly known that any second-order linear ODE \mathcal{E} is locally reduced to the free particle equation $x'' = 0$ by an equivalence transformation, so-called Arnold's transformation,

$$\bar{t} = \frac{\varphi^2(t)}{\varphi^1(t)}, \quad \bar{x} = \frac{x - \varphi^0(t)}{\varphi^1(t)}, \quad (3)$$

where φ^0 is a particular solution of the equation \mathcal{E} , φ^1 and φ^2 are linearly independent solutions of the corresponding homogenous equation, see, e.g., [1, p. 57] or [8]. In other words, the class (1) is a single orbit of its equivalence group G^\sim . Any class with this property is semi-normalized. We show this in detail. Consider two fixed equations \mathcal{E}_1 and \mathcal{E}_2 from the class (1) with $r = 2$ and a point transformation \mathcal{T} linking these equations. Let \mathcal{T}_1 and \mathcal{T}_2 be the projections of elements of G^\sim that map the equations \mathcal{E}_1 and \mathcal{E}_2 , respectively, to $x'' = 0$. Then the transformation $\mathcal{T}_0 := \mathcal{T}_2 \mathcal{T} \mathcal{T}_1^{-1}$ belongs to the point symmetry group of the equation $x'' = 0$. This implies the representation for \mathcal{T} , $\mathcal{T} = \mathcal{T}_2^{-1} \mathcal{T}_0 \mathcal{T}_1$. Roughly speaking, \mathcal{T} is the composition of equivalence transformations \mathcal{T}_1 and \mathcal{T}_2^{-1} and the symmetry transformation \mathcal{T}_0 of the equation $x'' = 0$. It is obvious that any transformation possessing such a representation maps the equation \mathcal{E}_1 to the equation \mathcal{E}_2 . The above representation can be rewritten as $\mathcal{T} = \tilde{\mathcal{T}} \hat{\mathcal{T}}$, where $\tilde{\mathcal{T}} = \mathcal{T}_2^{-1} \mathcal{T}_1$ is an equivalence transformation of the class (1) and $\hat{\mathcal{T}} = \mathcal{T}_1^{-1} \mathcal{T}_0 \mathcal{T}_1$ is a symmetry transformation of the equation \mathcal{E}_1 , which means that the class (1) is semi-normalized. \square

In what follows we consider the class (1) with $r \geq 3$. Although the projections of the equivalence group to the variable space in the case $r \geq 3$ coincide with that in the case $r = 2$, the corresponding equivalence groupoids have different structures.

Proposition 3. *The equivalence group G^\sim of the class (1), where $r \geq 3$, consists of the transformations whose projections to the variable space have the form (2). This group generates the entire equivalence groupoid of the class (1), i.e., the class (1) is normalized in the usual sense.*

Proof. In order to study admissible transformations in the class (1), we consider a pair of equations from this class, namely an equation of the form (1) and the equation

$$\tilde{x}^{(r)} + \tilde{a}_{r-1}(\tilde{t})\tilde{x}^{(r-1)} + \dots + \tilde{a}_1(\tilde{t})\tilde{x}' + \tilde{a}_0(\tilde{t})\tilde{x} = \tilde{b}(\tilde{t}), \quad (4)$$

and assume that these equations are connected by a point transformation \mathcal{T} of the general form

$$\tilde{t} = T(t, x), \quad \tilde{x} = X(t, x), \quad (5)$$

where the Jacobian $J = |\partial(T, X)/\partial(t, x)|$ does not vanish. At first we rewrite the derivatives with tildes in the equation (4) through the variables without tildes by using the following formula

$$\tilde{x}^{(k)} = \left(\frac{1}{DT} D \right)^k X,$$

where $D = \partial_t + x' \partial_x + x'' \partial_{x'} + \dots$ is the operator of total differentiation with respect to the variable t . After substituting the expressions for the tilde variables and derivatives into (4), we derive an equation in the untilde variables. It should be an identity on the manifold determined by (1) in the space of the variables t and x . The coefficient of $x'' x^{(r-1)}$ in this equation equals

$$-\frac{J}{(DT)^{r+2}} T_x \left(3 + \frac{(r-2)(r+3)}{2} \right) = 0,$$

and hence $T_x = 0$, i.e., the function T does not depend on the variable x . Taking into account this condition, we collect coefficients of $x' x^{(r-1)}$, which gives $r T_t^{-r} X_{xx} = 0$. This equation and the condition $J \neq 0$ imply that $X_{xx} = 0$, i.e., X is a linear function of x , $X = X^1(t)x + X^0(t)$. Therefore, the transformation \mathcal{T} has the form (2). The other determining equations obtained by splitting respect to the derivatives of x establish the relations between arbitrary elements of the initial and transform equations.

The transformation \mathcal{T} maps any equation from the class (1) to another equation from the same class. Hence the prolongations of the point transformations of the form (2) to the arbitrary elements a_{r-1}, \dots, a_0 and b constitute the equivalence group G^\sim . \square

Using parameterized families of transformations from the equivalence group G^\sim , we can gauge arbitrary elements of the class (1). For example, we can set $a_{r-1} = 0$. This gauge can be realized by the transformation

$$\bar{t} = t, \quad \bar{x} = \exp \left(\frac{1}{r} \int a_{r-1}(t) dt \right) x \quad (6)$$

from the group G^\sim , which maps the class \mathcal{L} onto the subclass \mathcal{L}_1 of equations in the *rational canonical form*

$$x^{(r)} + a_{r-2}(t)x^{(r-2)} + \dots + a_1(t)x' + a_0(t)x = b(t), \quad (7)$$

where we omitted bar over variables and arbitrary elements.

Namely, the canonical form (7) was used in [13, 19] for group classification in the framework of the infinitesimal approach.

Proposition 4. *The equivalence group G_1^\sim of the subclass \mathcal{L}_1 , where $r \geq 3$, consists of the transformations whose projections to the variable space have the form*

$$\tilde{t} = T(t), \quad \tilde{x} = C(T_t(t))^{\frac{r-1}{2}} x + X^0(t), \quad (8)$$

where T and X^0 are arbitrary smooth functions of t with $T_t \neq 0$, C is a nonzero constant and, in the real case with even r , the absolute value of T_t should be substituted instead of T_t . This group generates the equivalence groupoid of this subclass, i.e., it is normalized in the usual sense.

Proof. Suppose that the equation (7) and the equation (4) with $\tilde{a}_{r-1} = 0$ are connected by a point transformation \mathcal{T} . In view of the Proposition 3 this transformation has the form (2). We express the variables with tildes and the corresponding derivatives in terms of the variables and derivatives without tildes. After substituting the expressions for the tilde variables and derivatives into for (4) with $\tilde{a}_{r-1} = 0$, we collect terms containing the derivative $x^{(r-1)}$, which gives

$$r \frac{X^1}{T_t^r} \left(\frac{X_t^1}{X^1} - \frac{r-1}{2} \frac{T_{tt}}{T_t} \right) x^{(r-1)}.$$

The coefficient of $x^{(r-1)}$ vanishes if $X^1 = C(T_t)^{\frac{r-1}{2}}$, where $C = \text{const}$. Therefore, the point transformation \mathcal{T} has the form $\tilde{t} = T(t)$, $\tilde{x} = C(T_t(t))^{\frac{r-1}{2}} x + X^0(t)$ and the transformations of these form and prolonged to the arbitrary elements constitute the equivalence group G_1^\sim of the subclass \mathcal{L}_1 . \square

Transformations from G_1^\sim are parameterized by an arbitrary function $T = T(t)$ with $T_t \neq 0$. Hence we can set $a_{r-2} = 0$ in the equation (7) by a transformation from the group G_1^\sim , where the parameter-function T is a solution of the equation

$$T_{ttt}T_t - \frac{3}{2}T_{tt}^2 + \frac{12}{r(r^2-1)}a_{r-2}T_t^4 = 0.$$

This transformation links the subclass \mathcal{L}_1 with the subclass \mathcal{L}_2 of equations in the *Laguerre-Forsyth canonical form*

$$x^{(r)} + a_{r-3}(t)x^{(r-3)} + \cdots + a_1(t)x' + a_0(t)x = b(t). \quad (9)$$

Note that, in contrast to the transformation (6), the above transformation does not preserve the corresponding subclass of linear ODEs with constant coefficients.

Proposition 5. *The equivalence group G_2^\sim of the subclass \mathcal{L}_2 , where $r \geq 3$, consists of the transformations whose projections to the variable space have the form*

$$\tilde{t} = \frac{\alpha t + \beta}{\gamma t + \delta}, \quad \tilde{x} = \frac{C}{(\gamma t + \delta)^{r-1}}x + X^0(t), \quad (10)$$

where $\alpha, \beta, \gamma, \delta$ and C are arbitrary constants with $\alpha\delta - \beta\gamma \neq 0$ and $C \neq 0$, and X^0 is arbitrary smooth function of t . This group generates the equivalence groupoid of this subclass, i.e., it is normalized in the usual sense.

Proof. Let a point transformation \mathcal{T} links equations \mathcal{E} and $\tilde{\mathcal{E}}$ from the class (9). (We assume that all the values in the equation $\tilde{\mathcal{E}}$ are with tildes.) Thus, \mathcal{T} has the form (2). We express the derivatives of \tilde{x} with respect to \tilde{t} in terms of (t, x) and substitute these expressions into equation $\tilde{\mathcal{E}}$. Collecting the coefficients of $x^{(r-2)}$ gives the equation

$$\frac{X^1}{T_t^r} \left(-\frac{r(r-1)}{2} \frac{X_{tt}^1}{X^1} + \frac{r(r-1)^2}{2} \frac{X_t^1}{X^1} \frac{T_{tt}}{T_t} + \frac{r(r-1)(r-2)}{6} \frac{T_{ttt}}{T_t} - \frac{(r-2)(r^2-1)r}{8} \frac{T_{tt}^2}{T_t^2} \right) = 0.$$

Substituting the relation $X^1 = C(T_t)^{\frac{r-1}{2}}$ into this equation, we get the condition

$$\frac{T_{ttt}}{T_t} - \frac{3}{2} \left(\frac{T_{tt}}{T_t} \right)^2 = 0.$$

In other words, Schwarzian derivative of the function T vanishes, i.e., the function T is fractional linear,

$$T(t) = \frac{\alpha t + \beta}{\gamma t + \delta},$$

where $\alpha, \beta, \gamma, \delta$ are arbitrary constants with $\alpha\delta - \beta\gamma \neq 0$. We substitute the expression for T into (8) and obtain transformations that map any equation from the subclass \mathcal{L}_2 to another equation from the same subclass. Therefore, prolongations of these transformations to arbitrary elements form the equivalence group G_2^\sim of the subclass \mathcal{L}_2 . \square

Remark 1. In fact, the equivalence groups G^\sim , G_1^\sim and G_2^\sim of the classes \mathcal{L} , \mathcal{L}_1 and \mathcal{L}_2 are known in the literature for a long time, see, e.g., [23, Section 6] and [30, Section 4.1]. At the same time, we explicitly describe the associated equivalence groupoids by proving that these classes are normalized.

Remark 2. Denote the subclass of \mathcal{L} that consists of the r th order linear homogenous ODEs by $\hat{\mathcal{L}}$. The usual equivalence group of the subclass $\hat{\mathcal{L}}$ is obtained from the equivalence group G^\sim of the entire class \mathcal{L} by setting $X^0 = 0$ and neglecting the transformation component for b . Any admissible transformation \mathcal{T} between equations from the subclass $\hat{\mathcal{L}}$ has the form (2), where the ratio X^0/X^1 runs through the solution set of the initial equation. Combining the transformation \mathcal{T} with the induced transformation of the arbitrary elements and varying them, we obtain an element of the generalized extended equivalence group of $\hat{\mathcal{L}}$. The attributes ‘generalized’ and ‘extended’ jointly mean that this group consists of transformations whose components corresponding to variables may depend on arbitrary elements, and this dependence may be nonlocal. Therefore, the class of linear homogeneous equations is semi-normalized in the usual sense and normalized with respect to the generalized extended equivalence group. Similar statements are true for the subclasses $\hat{\mathcal{L}}_1$ and $\hat{\mathcal{L}}_2$ of homogeneous equations from \mathcal{L}_1 and \mathcal{L}_2 , respectively, and are known in the literature for classes of homogenous linear PDEs [28].

The above gauges of arbitrary elements of the class (1), which are commonly used, concern the leading coefficients a_{r-1} and a_{r-2} , but this is not a unique possibility. One can also gauge the lowest coefficients a_0 and a_1 . We can set $a_0 = 0$ in any equation from the class \mathcal{L} by an Arnold transformation

$$\tilde{t} = t, \quad \tilde{x} = \frac{x}{\varphi^1},$$

where $\varphi^1 = \varphi^1(t)$ is a nonzero solution of the corresponding homogeneous equation. As a result, we obtain the subclass \mathcal{A}_1 of the class \mathcal{L} that is constituted by the equations of the form

$$x^{(r)} + a_{r-1}(t)x^{(r-1)} + \dots + a_1(t)x^{(1)} = b(t). \quad (11)$$

Following the (eponymous) Arnold Principle³, we call this form the first Arnold canonical form.

Proposition 6. *The generalized extended group of the subclass \mathcal{A}_1 , where $r \geq 3$, consists of the transformations that act on the variable space as*

$$\tilde{t} = T(t), \quad \tilde{x} = \frac{x}{\psi^1(t)} + X^0(t), \quad (12)$$

where T and X^0 are arbitrary smooth functions of t with $T_t \neq 0$, and $\psi^1 = \psi^1(t)$ is a nonzero solution of the homogeneous equation associated with the initial one. This group generates the equivalence groupoid of this subclass, i.e., it is normalized in the generalized extended sense.

Proof. Let \mathcal{T} be a point transformation between equations \mathcal{E} and $\tilde{\mathcal{E}}$ of the form (11). Then the transformation \mathcal{T} is of the general form (2). As the equation $\tilde{\mathcal{E}}$ belongs to the subclass \mathcal{A}_1 , the function $\tilde{\psi}^1 \equiv 1$ is a solution of the homogeneous equation associated with $\tilde{\mathcal{E}}$. Hence $\psi^1 = 1/X^1$ is a solution of the homogeneous equation corresponding to \mathcal{E} . Therefore, the transformation \mathcal{T} is of the form (12). Since any transformation of this form can be applied to each appropriate equation from the subclass \mathcal{A}_1 , the prolongations of such transformations to the arbitrary elements constitute the generalized extended equivalence group of the subclass \mathcal{A}_1 . \square

Proposition 7. *The usual equivalence group $G_{\mathcal{A}_1}^\sim$ of the subclass \mathcal{A}_1 , where $r \geq 3$, consists of the transformations whose projections to the variable space have the form*

$$\tilde{t} = T(t), \quad \tilde{x} = Cx + X^0(t), \quad (13)$$

where T and X^0 are arbitrary smooth functions of t with $T_t \neq 0$ and C is an arbitrary nonzero constant.

³ The Arnold Principle states that if a notion bears a personal name, then this name is not the name of the discoverer. The Berry Principle extends the Arnold Principle by stating the following: the Arnold Principle is applicable to itself.

Proof. In view of Proposition 6 it is clear that the projection of any transformation from the group $G_{\mathcal{A}_1}^\sim$ to the variable space is of the form (12). Note that constant functions are solutions of any homogeneous equation from the subclass \mathcal{A}_1 . Therefore, the group $G_{\mathcal{A}_1}^\sim$ contains the transformations whose restrictions on the variables have the form (13). Moreover, only these transformations are in the group $G_{\mathcal{A}_1}^\sim$. Indeed, consider a transformation \mathcal{T} of the form (12) with $\psi^1 \neq \text{const}$. Then there exists a homogeneous equation from \mathcal{A}_1 that is not satisfied by the function ψ^1 . Hence the coefficient \tilde{a}_0 of the corresponding transformed equation is nonzero. This means that the transformation \mathcal{T} is not a projection of elements of the group $G_{\mathcal{A}_1}^\sim$. \square

Using the transformation of the form (3) with $\varphi^0 = 0$, we can set additionally $a_1 = 0$ in any equation from the class \mathcal{A}_1 . In this way the class \mathcal{A}_1 reduces to its subclass \mathcal{A}_2 that consists of the equations of the form

$$x^{(r)} + a_{r-1}(t)x^{(r-1)} + \dots + a_2(t)x^{(2)} = b(t), \quad (14)$$

called the second Arnold canonical form.

Proposition 8. *The generalized extended group of the subclass \mathcal{A}_2 , where $r \geq 3$, consists of the transformations whose projections to the variable space have the form*

$$\tilde{t} = \frac{\psi^2(t)}{\psi^1(t)}, \quad \tilde{x} = \frac{x}{\psi^1(t)} + X^0(t), \quad (15)$$

where $\psi^1(t)$ and $\psi^2(t)$ are linearly independent solutions of the homogeneous equation associated with the initial one, and $X^0(t)$ is an arbitrary smooth function of t . This group generates the equivalence groupoid of this subclass, i.e., it is normalized in the generalized extended sense.

Proof. Suppose that a point transformation \mathcal{T} connects two equations \mathcal{E} and $\tilde{\mathcal{E}}$ from the class \mathcal{A}_2 . Then the transformation \mathcal{T} has the form (12). Note that the function $\tilde{\psi}^2 = \tilde{t}$ is a solution of the homogeneous equation associated with $\tilde{\mathcal{E}}$. Hence the function $\psi^2(t) = \psi^1(t)T(t)$ is a solution of the homogeneous equation corresponding to \mathcal{E} , i.e., $T(t) = \psi^2(t)/\psi^1(t)$. As a result, the transformation \mathcal{T} is the form (15). As any transformation of this form maps appropriate equations from the class \mathcal{A}_2 to equations from the same class, the prolongations of such transformations to the arbitrary elements constitute the generalized extended equivalence group of the subclass \mathcal{A}_2 . \square

Proposition 9. *The usual equivalence group $G_{\mathcal{A}_2}^\sim$ of the subclass \mathcal{A}_2 , where $r \geq 3$, consists of the transformations whose projections to the variable space have the form*

$$\tilde{t} = \frac{\alpha t + \beta}{\gamma t + \delta}, \quad \tilde{x} = \frac{x}{\gamma t + \delta} + X^0(t), \quad (16)$$

where α, β, γ and δ are arbitrary constants with $\alpha\delta - \beta\gamma \neq 0$.

Proof. Any transformation from the group $G_{\mathcal{A}_2}^\sim$ belongs to the generalized extended group of the subclass \mathcal{A}_2 , and hence its projection to the variable space has the form (15). Since all linear functions of t are solutions of any homogeneous equation from the subclass \mathcal{A}_2 , then the group $G_{\mathcal{A}_2}^\sim$ contains the transformations whose restrictions on the variable space are of the form (16). We prove that there are no other transformations in the group $G_{\mathcal{A}_2}^\sim$, i.e., a transformation \mathcal{T} of the form (15) does not induce an equivalence transformation of the class \mathcal{A}_2 if at least one of the parameter-functions ψ^1 or ψ^2 is a nonlinear function of t . First consider the case where ψ^1 is nonlinear. We take a homogeneous equation \mathcal{E} from the subclass \mathcal{A}_2 that is not satisfied by ψ^1 . Then the corresponding transformed equation $\tilde{\mathcal{E}}$ possesses no constant solutions and hence its coefficient \tilde{a}_0 is nonzero. This means that the equation $\tilde{\mathcal{E}}$ does not belong to the subclass \mathcal{A}_2 , which gives the necessary statement. Now consider the complementary case, namely,

where the function ψ^1 is linear and the function ψ^2 is nonlinear. Then there is a homogeneous equation \mathcal{E} of the form (14) that is not satisfied by ψ^2 . As ψ^1 is a solution of the equation \mathcal{E} , the coefficient \tilde{a}_0 is equal to zero in the corresponding transformed equation $\tilde{\mathcal{E}}$. Moreover, the function $\tilde{\psi}^2 \equiv \tilde{t}$ is not a solution of the equation $\tilde{\mathcal{E}}$, so its coefficient \tilde{a}_1 is nonzero. Therefore, the equation $\tilde{\mathcal{E}}$ is not contained in the subclass \mathcal{A}_2 , which completes the proof. \square

Remark 3. By $\hat{\mathcal{A}}_1$ we denote the subclass consisting of homogeneous equations of the form (11), i.e., singled out from the class \mathcal{A}_1 by the constraint $b = 0$. The usual equivalence group of the subclass $\hat{\mathcal{A}}_1$ is obtained from the group $G_{\mathcal{A}_1}^\sim$ by setting $X^0 = 0$ and neglecting the transformation component for the arbitrary element b . Any admissible transformation \mathcal{T} in the subclass $\hat{\mathcal{A}}_1$ has the form (12), where the product $\psi^1 X^0$ runs through the solution set of the corresponding initial equation. Such transformations additionally prolonged to the arbitrary elements constitute the generalized extended equivalence group of $\hat{\mathcal{A}}_1$. Therefore, the subclass $\hat{\mathcal{A}}_1$ is normalized in the generalized extended sense, and in the usual sense it is even not semi-normalized. Similar statements are true for the subclass $\hat{\mathcal{A}}_2$ of homogeneous equations from \mathcal{A}_2 .

3 Group classification

Any inhomogeneous linear ODE can be reduced to the corresponding homogeneous equation by a transformation from the equivalence group G^\sim . Therefore, for solving the group classification problem for the class \mathcal{L} it suffices to solve the same problem for the subclass $\hat{\mathcal{L}}$ of homogeneous equations, which have the form (1) with $b = 0$.

As remarked earlier, any second-order linear ODE can be reduced to the elementary equation $x'' = 0$ (see, e.g., [15]), which admits the eight-dimensional Lie invariance algebra

$$\langle \partial_t, \partial_x, t\partial_t, x\partial_t, t\partial_x, x\partial_x, tx\partial_t + x^2\partial_x, t^2\partial_t + tx\partial_x \rangle.$$

This gives the exhaustive group classification of second-order linear ODEs.

Let $r \geq 3$. It is also a classical result by Lie [17, pp. 296–298] that the dimension of Lie invariance algebras for r th order ODEs with $r \geq 3$ is not greater than $r + 4$. Much later this result was partially reproved in [13, 19] only for linear ODEs.

Consider an r th order linear homogenous ODE \mathcal{E} . Due to linearity leading to the linear superposition principle, this equation admits the r -dimensional abelian Lie invariance algebra $\mathfrak{g}_a^\mathcal{E}$ spanned by the vector fields

$$\varphi^1(t)\partial_x, \varphi^2(t)\partial_x, \dots, \varphi^r(t)\partial_x, \tag{17}$$

where the functions $\varphi^i = \varphi^i(t)$, $i = 1, \dots, r$, form a fundamental set of solutions of the equation \mathcal{E} . By virtue of homogeneity, the equation \mathcal{E} also admits the one-parameter symmetry group of scale transformations generated by the vector field $x\partial_x$. Therefore, each equation \mathcal{E} from the subclass $\hat{\mathcal{L}}$ admits the $(r+1)$ -dimensional Lie invariance algebra

$$\mathfrak{g}_0^\mathcal{E} = \langle x\partial_x, \varphi^1(t)\partial_x, \varphi^2(t)\partial_x, \dots, \varphi^r(t)\partial_x \rangle. \tag{18}$$

Proposition 3 and Remark 2 imply that any Lie symmetry operator Q of the equation \mathcal{E} have the form $Q = \tau(t)\partial_t + (\xi^1(t)x + \xi^0(t))\partial_x$, where τ , ξ^1 and ξ^0 are smooth function on t , and ξ^0 is additionally a solution of \mathcal{E} . Hence the maximal Lie invariance algebra $\mathfrak{g}^\mathcal{E}$ of \mathcal{E} contains the subalgebra $\mathfrak{g}_0^\mathcal{E}$ as an ideal.

Now we carry out the group classification of linear ODEs in three different ways, which respectively involve the Laguerre–Forsyth canonical form (9), the rational canonical form (7) and Lie’s classification of realizations of finite-dimensional Lie algebras in the space of two variables.

The first way. As both the arbitrary elements a_{r-1} and a_{r-2} can be gauged to zero by equivalence transformations, the group classification of the class $\widehat{\mathcal{L}}$ can be further reduced to the group classification of its subclass $\widehat{\mathcal{L}}_2$ of homogeneous linear r th order ODEs in the Laguerre–Forsyth canonical form, which is singled out from $\widehat{\mathcal{L}}$ by the constraint $a_{r-1} = a_{r-2} = 0$. In other words, the class $\widehat{\mathcal{L}}_2$ consists of ODEs of the form

$$x^{(r)} + a_{r-3}(t)x^{(r-3)} + \dots + a_1(t)x' + a_0(t)x = 0. \quad (19)$$

The equivalence groupoid of the subclass $\widehat{\mathcal{L}}_2$ is generated by its generalized extended equivalence group. This $(r+4)$ -parameter group is constituted by the transformations whose projections to the variable space have the form (10), where the parameter-function $X^0 = X^0(t)$ is an arbitrary solution of the initial equation and, therefore, nonlocally depends on arbitrary elements. Up to transformations associated with the linear superposition principle and homogeneity, i.e., related to the Lie algebra $\mathfrak{g}_0^\mathcal{E}$, all admissible transformations within $\widehat{\mathcal{L}}_2$ are exhausted by transformations of the form (10), where $X^0 = 0$, $C = 1$ and the expression $\alpha\delta - \beta\gamma$ is equal to 1 or ± 1 in the complex or real case respectively. These transformations constitute the group isomorphic to the projective general linear group $\text{PGL}(2, \mathbb{F})$ of fractional linear transformations of t . Therefore, all possible Lie symmetry extensions within the class $\widehat{\mathcal{L}}_2$ are necessarily associated with subgroups of $\text{PGL}(2, \mathbb{F})$. In other words, the maximal Lie invariance algebra of the equation \mathcal{E} is a semidirect sum of the algebra $\mathfrak{g}_0^\mathcal{E}$ and a subalgebra of the realization of $\mathfrak{sl}(2, \mathbb{R})$ spanned by the vector fields $\mathcal{P} = \partial_t$, $\mathcal{D} = t\partial_t + \frac{1}{2}(r-1)x\partial_x$ and $\mathcal{K} = t^2\partial_t + (r-1)tx\partial_x$. Subalgebras of $\mathfrak{sl}(2, \mathbb{F})$ are well known (see, e.g., [24] or the appendix in the arXiv version of [26]). Up to internal automorphisms of $\mathfrak{sl}(2, \mathbb{F})$, a complete list of subalgebras of the above realization of $\mathfrak{sl}(2, \mathbb{F})$ is exhausted by the zero subalgebra $\{0\}$, the three one-dimensional subalgebras $\langle \mathcal{P} \rangle$, $\langle \mathcal{D} \rangle$ and $\langle \mathcal{P} + \mathcal{K} \rangle$, the two-dimensional subalgebra $\langle \mathcal{P}, \mathcal{D} \rangle$ and the entire realization $\langle \mathcal{P}, \mathcal{D}, \mathcal{K} \rangle$.

The zero subalgebra $\{0\}$ corresponds to the general case with no extension.

For one-dimensional extensions of (18) by the subalgebras $\langle \mathcal{P} \rangle$, $\langle \mathcal{D} \rangle$ and $\langle \mathcal{P} + \mathcal{K} \rangle$ the corresponding equations from the class (19) respectively take the forms

$$x^{(r)} + c_{r-3}x^{(r-3)} + \dots + c_1x' + c_0x = 0, \quad (20)$$

$$x^{(r)} + c_{r-3}t^{-3}x^{(r-3)} + \dots + c_1t^{-r+1}x' + c_0t^{-r}x = 0, \quad (21)$$

$$x^{(r)} + q_{r-3}(t)x^{(r-3)} + \dots + q_1(t)x' + q_0(t)x = 0, \quad (22)$$

where c_0, \dots, c_{r-3} are arbitrary constants,

$$q_{r-3}(t) = \frac{c_{r-3}}{(1+t^2)^3},$$

$$q_m(t) = \frac{c_m}{(1+t^2)^{r-m}} - \frac{(m+1)(r-m-1)}{(1+t^2)^{r-m}} \int (1+t^2)^{r-m-1} q_{m+1}(t) dt, \quad m = r-4, \dots, 0.$$

However, by the point transformations

$$\tilde{t} = \ln t, \quad \tilde{x} = xt^{-(r-1)/2} \quad \text{and} \quad \tilde{t} = \arctan t, \quad \tilde{x} = x(1+t^2)^{-(r-1)/2}$$

the maximal Lie invariance algebras of equations of the form (21) (known as the Euler–Cauchy equation, or just Euler’s equation) and the form (22) are reduced to the algebras looking as

$$\langle \partial_{\tilde{t}}, \tilde{x}\partial_{\tilde{x}}, \tilde{\varphi}^1(\tilde{t})\partial_{\tilde{x}}, \tilde{\varphi}^2(\tilde{t})\partial_{\tilde{x}}, \dots, \tilde{\varphi}^r(\tilde{t})\partial_{\tilde{x}} \rangle. \quad (23)$$

Moreover, the above transformations map these equations to constant-coefficients equations from the class (7), where $a_{r-2} = -\frac{1}{24}r(r^2 - 1)$ for (21) and $a_{r-2} = \frac{1}{6}r(r^2 - 1)$ for (22). Additionally scaling t , we can set $a_{r-2} = -1$ and $a_{r-2} = 1$, respectively. Thus, any equation from $\widehat{\mathcal{L}}_2$ that admits $(r+2)$ -dimensional Lie invariance algebra is equivalent to a homogeneous equation with

constant coefficients from the class (7), where $a_{r-2} = 0$, $a_{r-2} = -1$ and $a_{r-2} = 1$ for (20), (21) and (22), respectively.

If an equation from $\widehat{\mathcal{L}}_2$ possesses the $(r+3)$ -dimensional Lie invariance algebra

$$\langle \partial_t, t\partial_t + \frac{1}{2}(r-1)x\partial_x, x\partial_x, \varphi^1(t)\partial_x, \varphi^2(t)\partial_x, \dots, \varphi^r(t)\partial_x \rangle,$$

then it has the form $x^{(r)} = 0$ and hence its maximal Lie invariance algebra is $(r+4)$ -dimensional,

$$\langle \partial_t, t\partial_t + \frac{1}{2}(r-1)x\partial_x, t^2\partial_t + (r-1)tx\partial_x, x\partial_x, \varphi^1(t)\partial_x, \varphi^2(t)\partial_x, \dots, \varphi^r(t)\partial_x \rangle.$$

Note that, as in the above algebras the functions $\varphi^1, \dots, \varphi^r$ form a fundamental set of solutions of the elementary equation, we can choose $\varphi^i = t^{i-1}$, $i = 1, \dots, r$. Thus, there is no r th order linear ODE whose maximal Lie invariance algebra is $(r+3)$ -dimensional. Moreover, if such an equation admits $(r+4)$ -dimensional Lie invariance algebra, then it is similar to the elementary equation $x^{(r)} = 0$ with respect to a point transformation of the form (2).

The second way. Consider now the rational canonical form of equations from the class (1), i.e., the subclass \mathcal{L}_1 of ODEs of the form (7). Again, each equation \mathcal{E} from the subclass $\widehat{\mathcal{L}}_1$ consisting of homogeneous equations from \mathcal{L}_1 possesses the $(r+1)$ -dimensional Lie invariance algebra $\mathfrak{g}_0^\mathcal{E}$, which is an ideal of the maximal Lie invariance algebra $\mathfrak{g}^\mathcal{E}$ of \mathcal{E} . At the same time, in contrast to the Laguerre–Forsyth canonical form, the equivalence group G_1^\sim of \mathcal{L}_1 is parameterized by an arbitrary function $T = T(t)$. Proposition 4 and Remark 2 imply that the algebra $\mathfrak{g}^\mathcal{E}$ is contained in the algebra $\langle \mathcal{R}(\tau) \rangle + \mathfrak{g}_0^\mathcal{E}$, where $\mathcal{R}(\tau) = \tau(t)\partial_t + \frac{1}{2}(r-1)\tau_t(t)\partial_x$, the parameter τ runs through the set of smooth functions of t and plus denotes the sum of vector spaces. It is easy to see that $[\mathcal{R}(\tau^1), \mathcal{R}(\tau^2)] = \mathcal{R}(\tau^1\tau_t^2 - \tau^2\tau_t^1)$. Hence $\mathfrak{g}_1^\mathcal{E} := \langle \mathcal{R}(\tau) \rangle \cap \mathfrak{g}^\mathcal{E}$ is a (finite-dimensional) subalgebra of $\mathfrak{g}^\mathcal{E}$. Each operator $\mathcal{R}(\tau)$ is completely defined by its projection $\text{pr}_t\mathcal{R}(\tau)$ to the space of the variable t , $\text{pr}_t\mathcal{R}(\tau) = \tau(t)\partial_t$. In other words, the algebras $\mathfrak{g}_1^\mathcal{E}$ and $\text{pr}_t\mathfrak{g}_1^\mathcal{E}$ are isomorphic. Moreover, the corresponding projection of the equivalence group⁴ of the subclass $\widehat{\mathcal{L}}_1$ coincides with the group of all local diffeomorphisms in the space of the variable t . As a result, the group classification of the class $\widehat{\mathcal{L}}_1$ reduces to the classification of realizations of finite-dimensional Lie algebras by vector fields in the space of single variable t . The latter classification is well known and was done by S. Lie himself. A complete list of equivalence realizations on the line is exhausted by the algebras $\{0\}$, $\langle \partial_t \rangle$, $\langle \partial_t, t\partial_t \rangle$ and $\langle \partial_t, t\partial_t, t^2\partial_t \rangle$, which gives $\{0\}$, $\langle \mathcal{P} \rangle$, $\langle \mathcal{P}, \mathcal{D} \rangle$ and $\langle \mathcal{P}, \mathcal{D}, \mathcal{K} \rangle$ as possible Lie symmetry extensions within the class \mathcal{L}_1 . Here $\mathcal{P} = \mathcal{R}(1)$, $\mathcal{D} = \mathcal{R}(t)$ and $\mathcal{K} = \mathcal{R}(t^2)$ are the same operators as those in the first way. If the equation \mathcal{E} admits the two-dimensional extension $\langle \mathcal{P}, \mathcal{D} \rangle$, then it coincides with the elementary equation $x^{(r)} = 0$, which possesses the three-dimensional extension $\langle \mathcal{P}, \mathcal{D}, \mathcal{K} \rangle$. This is why the two-dimensional extension is improper. Finally, we have three inequivalent extensions in the class \mathcal{L} , which are the general case with no extension, general constant-coefficients equations admitting the one-dimensional extension $\langle \mathcal{P} \rangle$, and the elementary equation $x^{(r)} = 0$ possessing the three-dimensional extension $\langle \mathcal{P}, \mathcal{D}, \mathcal{K} \rangle$.

The third way. The group classification of the entire class $\widehat{\mathcal{L}}$ can also be obtained directly from Lie’s classification of realizations of finite-dimensional Lie algebras in the spaces of two real or complex variables [15, 16]. A modern treatment of these results was presented, e.g., in [9, 22]. In order to solve the group classification problem for the class $\widehat{\mathcal{L}}$, from Lie’s list of realizations we select candidates for the maximal invariance algebras of nonequivalent equations from $\widehat{\mathcal{L}}$. All candidates should satisfy the following obvious properties, which are preserved by any point transformations: The maximal invariance algebra $\mathfrak{g}^\mathcal{E}$ of each r th order linear ODE \mathcal{E} ($r \geq 3$) contains the $(r+1)$ -dimensional almost abelian ideal $\mathfrak{g}_0^\mathcal{E}$. More precisely, the ideal $\mathfrak{g}_0^\mathcal{E}$

⁴The explicit indication what kind the equivalence group is of is not essential here due to each of the following arguments: (i) The projections of the usual and generalized extended equivalence groups of the subclass \mathcal{L}_1 to the space of the variable t are the same. (ii) The subclass \mathcal{L}_1 is semi-normalized in the usual sense.

is the semidirect sum of an r -dimensional abelian ideal $\mathfrak{g}_a^\mathcal{E}$ of whole algebra $\mathfrak{g}^\mathcal{E}$ and one more vector field whose adjoint action on the ideal $\mathfrak{g}_a^\mathcal{E}$ is the identity operator. Moreover, $\text{rank } \mathfrak{g}_0^\mathcal{E} = 1$. The above properties are satisfied by realization families 21, 23, 26 and 28 from [9, Table 1] (or realization families 49, 51, 54 and 56 from [22, Table 1.1], respectively). As in the previous two ways, among r th order linear ODEs, only the elementary equation $x^{(r)} = 0$ admits the third realization. At the same time, this equation also possesses the fourth realization, which is of greater dimension than the third one. This is why the third realization should be neglected. Given two proper candidates that are equivalent with respect to a point transformation \mathcal{T} and whose $(r+1)$ -dimensional almost abelian ideals are of the form (18), the transformation \mathcal{T} should have the form (2), where X^0 is a linear combinations of the parameters-functions $\varphi^1, \dots, \varphi^r$ of the initial realization. This is why the point equivalence within the chosen families of realizations well conforms with the point equivalence of linear ODEs.

As a result, we have reproved the following assertion (see, e.g., [13, 19, 23, 30, 40]):

Proposition 10. *The dimension of the maximal Lie invariance algebra $\mathfrak{g}^\mathcal{E}$ of an r th order ($r \geq 3$) linear ODE takes a value from $\{r+1, r+2, r+4\}$. If $\dim \mathfrak{g}^\mathcal{E} > r+1$, then the equation \mathcal{E} is similar to a linear ODE with constant coefficients. In the case $\dim \mathfrak{g}^\mathcal{E} = r+4$ the equation \mathcal{E} is reduced by a point transformation of the form (2) to the elementary equation $x^{(r)} = 0$.*

Recall that in [13, 19] this assertion was proved by using the classical infinitesimal approach, which involved cumbersome calculations. A more elegant approach for solving the group classification problem for the class (1), which is based on the construction of relative invariants, was discussed in [2, 23, 30]. The similar problem on the classification of linear ODEs up to contact transformations as well as the associated equivalence problem was considered in detail in [38, 39, 40].

4 Conclusion

In this paper we exhaustively describe the equivalence groupoid of the class \mathcal{L} of r th order linear ODEs as well as equivalence groupoids of its subclasses \mathcal{L}_1 , \mathcal{L}_2 , \mathcal{A}_1 and \mathcal{A}_2 associated with the rational, Laguerre–Forsyth, first and second Arnold canonical forms, i.e., the classes (1), (7), (9), (11) and (14) respectively.

The case $r = 2$ is considered separately. We shown that, roughly speaking, the equivalence groupoid of the class \mathcal{L} with $r = 2$ is generated by transformations from the equivalence group of the class \mathcal{L} and transformations from the point symmetry group of the equation $x'' = 0$, see Proposition 2.

For $r \geq 3$, the equivalence groupoids of the classes \mathcal{L} , \mathcal{L}_1 and \mathcal{L}_2 are generated by the corresponding (usual) equivalence groups; in other words, each of these classes is normalized in the usual sense, see Propositions 3–5. This allows us to classify Lie symmetries of liner ODEs using algebraic tools in three different ways. The purpose of the presentation of various ways for carrying out the known classification is to demonstrate advantages and disadvantages of each of them, which is important, e.g., to effectively apply the algebraic approach to systems of linear ODEs. Thus, the classification based on the Laguerre–Forsyth canonical form, which is associated with a maximal gauge of arbitrary elements, is just reduced to the classification of subalgebras of the algebra $\mathfrak{sl}(2, \mathbb{R})$, which is finite-dimensional (more precisely, three-dimensional). Using the rational canonical form leads to involving the classification of all possible realization of finite-dimensional Lie algebras on the line. At the same time, the single classification case of constant-coefficient equations in the rational canonical form is split in the Laguerre–Forsyth canonical form into three cases, and two of them are related to variable-coefficient equations. If we neglect the possibility of gauging arbitrary elements and consider general linear ODEs, we need to classify specific realizations of specific Lie algebras in the space of two variables.

The structure of the equivalence groupoids of the subclasses \mathcal{A}_1 and \mathcal{A}_2 is more complicated than that for \mathcal{L} , \mathcal{L}_1 and \mathcal{L}_2 . More precisely, the subclasses \mathcal{A}_1 and \mathcal{A}_2 are normalized in the generalized extended sense but not in the usual sense and, moreover, the corresponding subclasses of homogenous equations are not semi-normalized in the usual sense. This is why the subclasses \mathcal{A}_1 and \mathcal{A}_2 are not enough usable for the group classification of the class \mathcal{L} .

In contrast to single linear ODEs, results concerning group properties of normal systems of second-order linear ODEs are very far from complete, not to mention general systems of linear ODEs; see a more detailed discussion in [4]. Only recently the group classification problem of systems of second-order linear ODEs with commuting constant coefficients matrices was considered for various particular cases of the number of equations (two, three or four) and the structure of the coefficient matrices in a series of papers [5, 6, 20, 35] and was then exhaustively solved in [4]. In spite of a number of publications on the subject, the group classification of systems of linear second-order ODEs with noncommuting constant coefficients matrices or with general nonconstant coefficients was carried out only for the cases of two and three equations [21, 32, 36]. The consideration of a greater number of equations or equations of higher and different orders within the framework of the classical infinitesimal approach or the theory of relative invariants, requires cumbersome computations. Hence there is a demand for the development of new more powerful algebraic and geometric tools, which, for instance, involve deep investigation of associated equivalence groupoids and other related algebraic structures. Our work in this direction is in progress and will be a subject of a forthcoming paper.

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